Static Analysis of Composite Laminated Beam By First Order Shear Deformation Theory

B.R.Barhate, Dr. U.P. Waghe

***Abstract*— Various displacement-based theories for laminated beams have been developed. Two approaches are usually adopted, one is without considering shear deformation and other is with shear deformation effect in the beam. Static analysis of composite laminated beam is presented in this work. Simply supported composite laminated beams are examined. Different values of aspect ratio of the beams, subjected to bending are considered for the analysis. Classical beam models, such as Euler–Bernoulli's and Timoshenko's, are obtained as particular cases. The longitudinal normal displacement and corresponding normal stress are analysed for the composite laminated beam. Results are validated in terms of accuracy and computational costs with available solution.**

***Index Terms*— Analysis, composite laminated beam, EBT, TBT.**

I. INTRODUCTION

A beam is a structural member which is subjected to transverse loads causing bending in addition with stretching. The cross sectional dimensions are much smaller than its length. Fiber-reinforced composite laminates are widely used in various engineering structures due to their high specific strength and high specific stiffness. Composite materials are being used increasingly in many civil and other Engineering applications such as aeronautical, aerospace, automobile and under water structures, due to their attractive properties like strength, stiffness and lightness The high stiffness-to-weight ratio coupled with the flexibility in the selection of lamination scheme that can be tailored to match the design requirements. This may be attributed to the fact that composites have high strength to weight ratio, stiffness to weight ratio, suitable thermal and electrical properties and low maintenance cost, etc.

On the other side, there nature more complex than classical materials due to wider number of parameters (such as anisotropy, geometry, material of the fibres and matrix and stacking sequence) govern their behavior. In addition, beam structures play an important role in many engineering fields. Highly accurate mechanical models are required to effectively describe the mechanics of composite structures. Researchers devoted lot of work to set a good method capable to describe the displacements and stresses of laminated beams subject to transversal loadings. . This is due to the fact that classical laminated beam theories, based on Euler–Bernoulli model, are unable to predict the behavior of deep beams made with anisotropic materials. The main motivation is related to the fact that transverse shear effects are disregarded. Timoshenko developed a first-order shear deformation theory.

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B. Mechanics of Laminated Composites

A greatest advantage of a laminated composite is the freedom to select the precise form of the materials (fibres and matrix), orientation and stacking sequence for tailoring the material to suit specific structural requirements. The task of analysis and design of a fibre reinforced composite structure is considerably more difficult than that of a metal structure, primarily due to the variation in its properties in different directions and number of layers with different properties. In the beginning, elasticity solutions based on the elastic theory were sought by some researchers. The approached used to solve the elastic problem include use of Airy's stress functions, solving partial differential equations, Saint Venant's Principle, etc.

The exact elasticity solutions of laminated beams have been presented by Lekhnitskii [1] using Airy's stress polynomial functions. In the analysis, each layer was assumed to be orthotropic, especially in the plane of bending. Schile [2] presented an elasticity solution for beams with variable modulus of elasticity and Poisson's ratio in the transverse direction. The solution was not restrictive in terms of assumed interfacial boundary conditions.

A review on different refined shear deformation theories for the analysis of isotropic and laminated beams was presented by Ghugal and Shimpi [3] for both equivalent single layer (ESL) and layer-wise (LW) models were compared. Their benefits and deficiencies were also discussed. Matsunaga [4] presented a global higher-order theory to compute displacements and stresses in composite beams subjected to transverse loadings. In that work, axial stresses were calculated through the constitutive relations, whereas transverse stresses were determined by integration of the three-dimensional equilibrium equations. A layer-wise trigonometric shear deformation theory (LTSDT-I) was presented by Shimpi and Ghugal [5]. They analysed two layered cross-ply beams by using the first-order shear deformation theory for each layer. The transverse shear stress was obtained through the constitutive relations. A unified quassi-3D HSDT analysis done by Mantari and Canales [9]. A static stress analysis of laminated composite cross ply beam is presented in this work. Classical beam models, such as Euler-Bernoulli's and Timoshenko beam are obtained as particular cases. Simply supported cross ply beams are analysed under transverse loading. Plane stress condition are

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used to approximate 2D problem in 1D state. Displacements and stresses are the concerned quantities. Results are validate to exact or other available solutions. The analysis were carried by using the programming software MATLAB to solve algebraic simultaneous equations.
The different theories used in the analysis of Composites are:

C. Euler-Bernoulli Beam Theory (EBT)

This theory also known as elementary beam theory, which is based on the assumption that planes initially normal to the mid-plane remain plane and normal to the mid-surface after bending leads to high percentage of error in the analysis of anisotropic beams due to neglect of shear deformation. This theory also known as classical beam theory. The Bernoulli beam is named after Jacob Bernoulli, who made the significant discoveries Euler and Daniel Bernoulli were the first to put together a useful theory circa. In Euler – Bernoulli beam theory, shear deformations are neglected, and plane sections remain plane and normal to the longitudinal axis. Euler-Bernoulli beam theory, is a simplification of the linear theory of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams. Euler-Bernoulli beam theory does not account for the effects of transverse shear strain. As a result it under predicts deflections and over predicts natural frequencies. For thin beams (beam length to thickness ratios of the order 20 or more) these effects are of minor importance. For thick beams, however, these effects can be significant. More advanced beam theories such as the Timoshenko beam theory have been developed to account for these effects. This theory is used for thin beams. For thick beams the theory needs some modifications to include the effect of transverse shear. Refined shear deformation theories are needed. When this theory is used for the analysis of laminated beams, deflections are underestimated and natural frequencies and buckling loads are overestimated. This is consequence of neglecting transverse shear deformation in EBT.

D. Timoshenko Beam Theory (TBT)

This theory is an improvement over elementary theory of beam and based upon kinematics it is known as first order shear deformation theory. TBT was developed by Stephen Timoshenko in the beginning of the 20th century. Timoshenko's theory of beams constitutes an improvement over the EBT. The difference between the Timoshenko beam and the Bernoulli beam is that the former includes the effect of the shear stresses on the deformation. Timoshenko showed that the effect of shear is much greater than that of rotary inertia for transverse vibration of prismatic beams. TBT is First order shear deformation theory.

A constant shear over the beam height is assumed. It is also said that the Timoshenko's beam theory is an extension of the EBT to allow for the effect of transverse shear deformation. TBT relaxes the normality assumption of plane sections that remain plane and normal to the deformed centerline. For example, in dynamic case, Timoshenko's theory incorporates shear and rotational inertia effects and it will be more accurate for not very slender beam.

The governing differential equations obtained are very comprehensive, covering and extending the current models for the problems that are based on Euler–Bernoulli beam theory.

II. ANALYTICAL MODELLING OF CROSS PLY LAMINATED BEAM

A. Beam under consideration

A composite layered beam of uniform thickness is considered. The plan dimension of beam is (a x b) where 'a' is along span and 'b' is width of beam. Thickness is '*h*'. The top surface of the beam is loaded with transversely distributed load, under such a condition that the beam domain is in a 2D state of plane-stress in x-z plane. The beam is simply supported on the longitudinal edge as shown in Fig.1

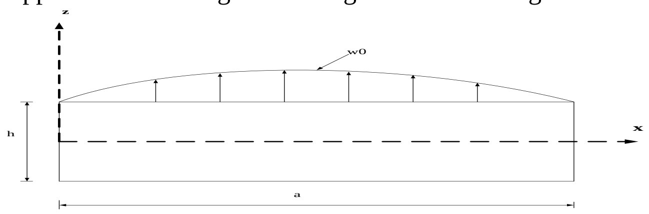

Fig.1. Beam under plane stress condition

B. Theoretical displacement field

A complete analytical formulation and solution for a narrow laminated beam simply supported along '*x*' axis is presented. The geometry of the narrow laminated beam is such that the side '*a*' is along '*x*' axis and side '*b*' is on '*y*' axis, which is assumed to be negligible. The thickness of the narrow laminated beam is denoted by '*h*' and is coinciding with '*z*' axis. The reference mid-plane of the narrow laminated beam is at *h/2* from top or bottom surface of the laminate as shown in the Figure.1. In narrow beam problem, the width dimension (along *y* direction) is very small as compared to other dimensions (along *x* and *z* directions). In such problems, the stresses along *y* direction are very small as compared to *x* and *z* directions and can be neglected. Then problem is assumed to be in two-dimensional and in a state of plane stress. Neglecting the stresses along *y* direction

The stress-strain relationship for a two-dimensional orthotropic body under plane stress condition can be state as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{13} & 0 \\ \bar{Q}_{13} & \bar{Q}_{33} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix}$$

The 2D equation of equilibrium are :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0$$

$$\frac{\partial \sigma_Z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + B_z = 0$$

C. Governing equations

The bending moment equation as

$$M_x = \int_{-h/2}^{+h/2} z\sigma_x dz$$

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz} dz$$

$$w = w_0 \sin\left(\frac{\pi x}{a}\right),$$

$$u = z\theta_x \cos\left(\frac{\pi x}{a}\right),$$

D. Solution procedure

A single layered orthotropic material (0^0) simply supported beam is considered in this example. The beam is subjected to single sinusoidal loading on the top edge. The normalized mid-plane transverse displacement $(\bar{w})$, longitudinal normal stress $(\bar{\sigma}_x)$ and transverse shear stress $(\bar{\tau}_{xz})$ for different length to depth ratios are presented in Table.1. Results obtained through the present analysis under plane stress conditions have been compared with Exact solution by Kant et al.[6] and others [4]. The following sets of data are used in obtaining numerical results.

$E_1 / E_2 = 25$, $E_1 = 25$GPa, $E_2 = E_3 = 1$GPa, $G_{12} = G_{23} = 0.25$, $G_{13} = 0.5$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25, \quad \nu_{21} = 0.01, \nu_{32} = 0.25$$

$$\nu_{31} = \left(\frac{E_3}{E_1}\right)\nu_{13}$$

III. NUMERICAL RESULTS:

The results are compared as follows:

Table:1 Comparison of normalized transverse displacement $(\bar{w})$, in-plane normal stress $(\bar{\sigma}_x)$ of simply supported orthotropic (0^0) beams.

The results are compared as follows:

a/h		[EBT][1]	[TBT][2]	[EXACT][3]	[HSDT][4]
4	w	0.4913	2.0107 [3.65%][3]	1.9509	1.7414 [10.71%][3]
	σ_x	0.6078	0.6078 [32.67%][3]	0.9028	0.5 [44.61%][3]
10	w	0.4913	0.7344 [0.15%][3]	0.7333	-
	σ_x	0.6078	0.6078 [7.48%][3]	0.6570	-
20	w	0.4913	0.5521 [0.19%][3]	0.5532	0.5429 [1.86%][3]
	σ_x	0.6078	0.6078 [2.01%]	0.6203	0.5 [19.39%][3]
30	w	0.4913	0.5183 [0.26%]	0.5197	-
	σ_x	0.6078	0.6078 [0.91%][3]	0.6134	-
40	w	0.4913	0.5065 [0.27%][3]	0.5079	0.5054 [0.49%][3]
	σ_x	0.6078	0.6078 [0.52%][3]	0.6110	0.5 [18.61%][3]
50	w	0.4913	0.5010 [0.23%][3]	0.5024	-
	σ_x	0.6078	0.6078 [0.34%][3]	0.6099	-
100	w	0.4913	0.4937	-	-
	σ_x	0.6078	0.6078	-	-

[EBT][1] - Present Analysis by Euler-Bernoulli's theory (EBT)
[TBT][2] - Present Analysis by Timoshenko beam theory (TBT)
[EXACT][3] - Kant et al. (2007)
[HSDT][4] - Matsunaga (2002)

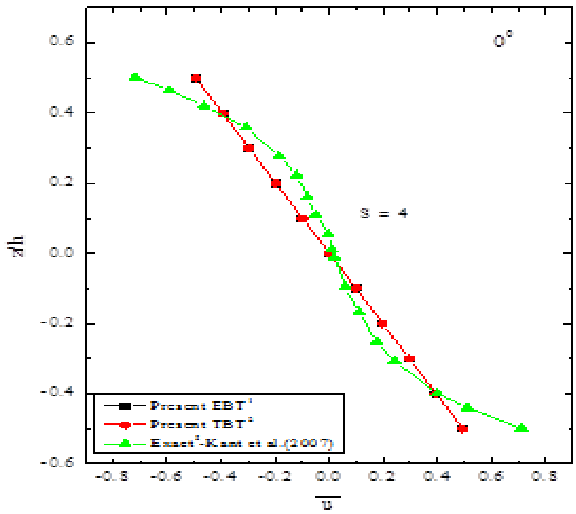

Fig.2.Variation of normalised longitudinal displacement $(\bar{u})$ through thickness of a simply supported orthotropic (0^0) beam under plane stress conditions

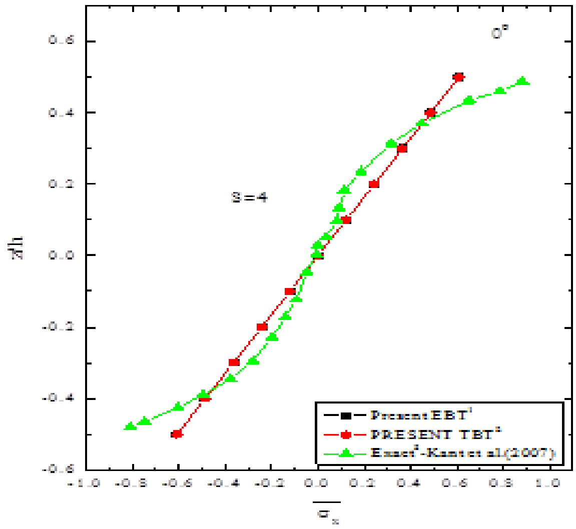

Fig.3.Variation of normalised longitudinal normal stress $(\bar{\sigma}_x)$through thickness of a simply supported orthotropic (0^0)beam under plane stress conditions

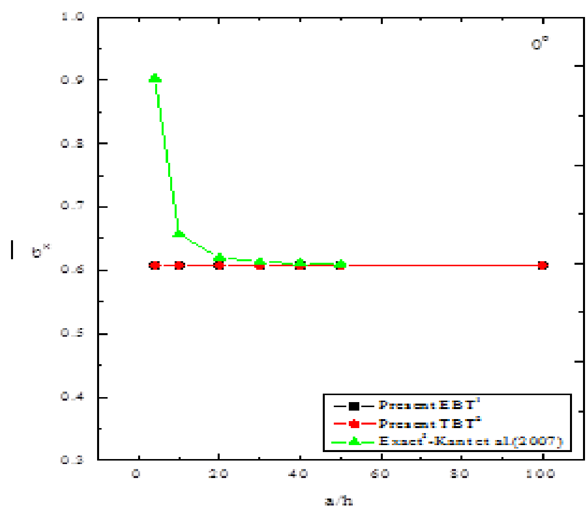

Fig.4. Variation of normalised longitudinal normal stress $(\bar{\sigma}_x)$ through aspect ratio of a simply supported orthotropic (0^0) beam under plane stress conditions

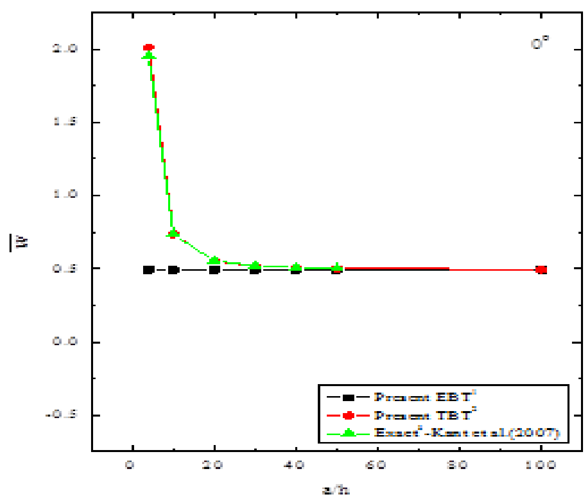

Fig.5. Variation of normalised transverse displacement $(\overline{w})$ *through aspect ratio of a simply supported orthotropic* (0^0) *beam under plane stress condition*

A discussion on the static analysis of single layer orthotropic simply supported beam is presented here.

Table 1 shows the comparison between present EBT and TBT results with Exact analysis of Kant et al[6]. The non dimensionalized quantities by present TBT are in good agreement with exact analysis. Fig 2-5 shows the graphical comparison of present through thickness quantities with others

IV. CONCLUSION

1. Static stress analysis is carried out by EBT and TBT. Examples of isotropic, orthotropic and laminates are considered.
2. Numerical results are compared with other theories such as HOST, Mixed FEM and Exact solution.
3. A complete analytical set is developed for EBT and TBT to find displacement and stress quantities and it is seen that TBT is an improvement over EBT because of shear deformation effect.
4. The improvement is observed in TBT over EBT for a/h=4, but again in-plane normal stress $(\bar{\sigma}_x)$ remains constant for all a/h ratio.
5. The variation of in-plane displacement$(\overline{u})$, in-plane normal stress $(\bar{\sigma}_x)$ through thickness of beam is linear, whereas nonlinear variation in exact solution.
6. As a/h ratio increases the EBT and TBT results for transverse displacement $(\overline{w})$ and in-plane normal stress $(\bar{\sigma}_x)$ converges after a/h=40.
7. Finally it conclude that the TBT is an improvement because it involve shear deformation effect Based on First order shear deformation

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